Hopf algebras and the combinatorics of connected graphs in quantum field theory

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Abstract

In this talk, we are concerned with the formulation and understanding of the combinatorics of time-ordered n-point functions in terms of the Hopf algebra of field operators. Mathematically, this problem can be formulated as one in combinatorics or graph theory. It consists in finding a recursive algorithm that generates all connected graphs in their Hopf algebraic representation. This representation can be used directly and efficiently in evaluating Feynman graphs as contributions to the n-point functions.

Recently, it was realized that the Hopf algebra structure of the algebra of field operators $\mathsf{S}(V)$ (with the normal or with the time-ordered product) can be fruitfully exploited. In particular, the Laplace Hopf algebra created by Rota *et al.* [4, 7] was generalized to provide an algebraic tool for combinatorial problems of quantum field theory [1]. In addition, Sweedler's Hopf algebra cohomology [12] and the Drinfel'd twist [5] were used to show that different products of the algebra of field operators are related by Drinfel'd

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twists and that interactions correspond to 2-cocycles [2]. Subsequently, the time-ordered Hoft algebra of field operators was used to define an algebraic representation of graphs [10, 11]. That is, every graph with v vertices was associated with an element of $\mathsf{S}(V)^{\otimes v}$, the v-fold tensor product of $\mathsf{S}(V)$. This representation allowed to derive simple algebraic relations between complete, connected and 1-particle irreducible (1PI) n-point functions [10], and to express a connected n-point function in terms of its loop order contributions [11]. The basic structure is an algorithm to recursively generate all weighted connected graphs. That is, each graph is generated together with a scalar that will correspond to the inverse of its symmetry factor. This algorithm is amenable to direct implementation and allows efficient calculations of connected graphs as well as their values as Feynman graphs. In particular, all results apply both to bosonic and fermionic fields. Moreover, this Hopf algebraic approach to express the connected Green functions in terms of the 1PI ones, was recently generalized to many-body physics [3].

This paper reviews the main results of [10, 11]. Section 1 recalls the Hopf algebra structure of the algebra of quantum field operators. Section 2 describes the Hopf algebraic representation of graphs and gives some examples. Section 3 sketches the algorithmic construction of connected graphs as well as the interpretation in terms of Feynman graphs and n-point functions. Section 4 focuses on expressing the relation between connected and 1PI n-point functions in a completely algebraic language.

1 Field operator algebra as a Hopf algebra

We briefly recall the Hopf algebra structure of the time-ordered field operator algebra. A more extensive discussion adapted to the present context, can be found in [10].

Let V denote the vector space of linear combinations of elementary field operators $\phi(x)$, where x denotes field operator labels. Let $\mathsf{S}(V) = \bigoplus_{v=0}^{\infty} V^v$ denote the free, unital, commutative algebra generated by all time-ordered products of field operators. In particular, $\mathsf{S}(V)$ is also a Hopf algebra [6, 8, 9]. Therefore, $\mathsf{S}(V)$ is equipped with a linear map $\Delta : \mathsf{S}(V) \to \mathsf{S}(V) \otimes \mathsf{S}(V)$, called coproduct, which is coassociative: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. This is defined on V by $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\Delta(\phi(x)) = \phi(x) \otimes \mathbf{1} + \mathbf{1} \otimes \phi(x)$, and extended to the all of $\mathsf{S}(V)$ due to the compatibility with the product. The coproduct may be interpreted as an operation to split a product of field operators into two parts in all possible ways. Moreover, on $\mathsf{S}(V)$ the counit $\epsilon : \mathsf{S}(V) \to \mathbb{C}$ is defined by $\epsilon(\mathbf{1}) = 1$ and $\epsilon(\phi(x_1) \cdots \phi(x_n)) = 0$ for $n > 0$.

The characterizing property of the counit is the equality $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$. Finally, to meet the requirements of a Hopf algebra, there is an antipode map $S : \mathsf{S}(V) \to \mathsf{S}(V)$ defined on $\mathsf{S}(V)$, by $S(\phi(x_1) \ldots \phi(x_n)) = (-1)^n \phi(x_1) \ldots \phi(x_n)$.

Now, let $G^{(n)}$, $G_c^{(n)}$, $G_{1\mathrm{PI}}^{(n)}$ and $\hat{G}_{1\mathrm{PI}}^{(n)}$ denote complete, connected, 1PI and modified 1PI[1] n-point functions, respectively. Moreover, let $\mathcal{V}$ denote vertex functions. The ensemble of time-ordered n-point functions of a given type, determine maps $\mathsf{S}(V) \to \mathbb{C}$:

$$\rho(\phi(x_1) \cdots \phi(x_n)) := G^{(n)}(x_1, \ldots, x_n) \,,$$
$$\sigma(\phi(x_1) \cdots \phi(x_n)) := G_c^{(n)}(x_1, \ldots, x_n) \,,$$
$$\tau(\phi(x_1) \cdots \phi(x_n)) := G_{1\mathrm{PI}}^{(n)}(x_1, \ldots, x_n) \,,$$
$$\hat{\tau}(\phi(x_1) \cdots \phi(x_n)) := \hat{G}_{1\mathrm{PI}}^{(n)}(x_1, \ldots, x_n) \,,$$
$$\nu(\phi(x_1) \cdots \phi(x_n)) := \mathcal{V}(x_1, \ldots, x_n) \,.$$

The assumption that all 1-point functions vanish means that $\rho(\phi(x)) = \sigma(\phi(x)) = \tau(\phi(x)) = \hat{\tau}(\phi(x)) = 0$. Moreover, the 0-point functions read as $\rho(\mathbf{1}) = 1$, $\sigma(\mathbf{1}) = \tau(\mathbf{1}) = \hat{\tau}(\mathbf{1}) = 0$. Besides, the 2-point function $\hat{\tau}(\phi(x)\phi(y))$ vanishes by construction.

2 A Hopf algebraic representation of graphs

We study the correspondence between graphs and elements of $\mathsf{S}(V)^{\otimes v}$ given in [10, 11].

Let G_F^{-1} denote the inverse Feynman propagator given by

$$\int \mathrm{d}y \, G_F(x, y) G_F^{-1}(y, z) = \delta(x, z) \,.$$

We consider the following formal operators defined on $\mathsf{S}(V)^{\otimes v}$:

$$R_{i,j} := \int \mathrm{d}x \, \mathrm{d}y \, G_F^{-1}(x, y) \, (\mathbf{1}^{\otimes i-1} \otimes \phi(x) \otimes \mathbf{1}^{\otimes j-i-1} \otimes \phi(y) \otimes \mathbf{1}^{\otimes v-j}), \quad (1)$$

where the field operators $\phi(x)$ and $\phi(y)$ are inserted at the positions i and j, respectively, with $i \neq j$; and

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[1]Recall that the connected n-point functions G_c are expressible in terms of $\hat{G}_{1\mathrm{PI}}$ as a sum over all tree graphs whose vertices have valence at least three, by associating a connected propagator $G_c^{(2)}$ to every edge.

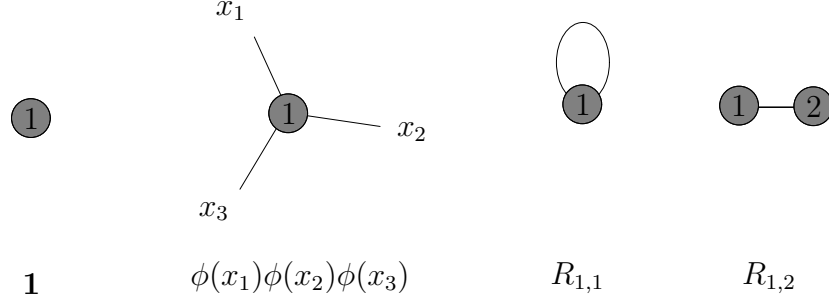$$\mathbf{1} \qquad\qquad \phi(x_1)\phi(x_2)\phi(x_3) \qquad\qquad R_{1,1} \qquad\qquad R_{1,2}$$

Figure 1: Examples of the correspondence between vertices and edges, and elements of $\mathsf{S}(V)^{\otimes v}$.

$$R_{i,i} := \int \mathrm{d}x \, \mathrm{d}y \, G_F^{-1}(x,y) \left(\mathbf{1}^{\otimes i-1} \otimes \phi(x)\phi(y) \otimes \mathbf{1}^{\otimes v-i} \right). \qquad (2)$$

The R-operators are employed in establishing a correspondence between graphs with v vertices and certain elements of $\mathsf{S}(V)^{\otimes v}$, the v-fold tensor product of $\mathsf{S}(V)$. Namely,

- a tensor factor in the ith position corresponds to a vertex numbered i, with $i = 1, \ldots, v$;

- a product $\phi(x_1)\cdots\phi(x_n)$ in a given tensor factor corresponds to external edges (of the associated vertex) whose end points are labeled by $x_1, \ldots, x_n$;

- the element $R_{i,j} \in \mathsf{S}(V)^{\otimes v}$ defined by (1), corresponds to an internal edge connecting the vertices i and j;

- the element $R_{i,i} \in \mathsf{S}(V)^{\otimes v}$ for $1 \leq i \leq v$ defined by (2), corresponds to an internal edge connecting the vertex i to itself (i.e., a self-loop).

Figure 1 shows examples of the correspondence between vertices, external edges and internal edges, and elements of $\mathsf{S}(V)^{\otimes v}$.

Combining several internal edges and their products with external edges by multiplying the respective expressions in $\mathsf{S}(V)^{\otimes v}$, allows to build arbitrary graphs with v vertices. Figure 2 shows some examples. Moreover, applying the vertex functions ν to each tensor factor yields precisely the value of the respective graph as a Feynman graph.
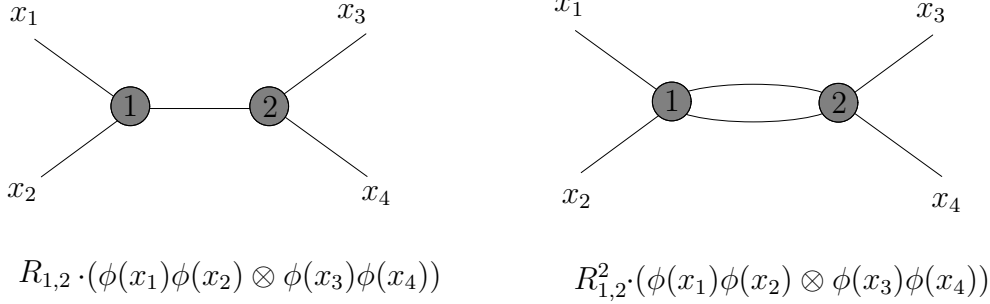
$$R_{1,2} \cdot (\phi(x_1)\phi(x_2) \otimes \phi(x_3)\phi(x_4)) \qquad R_{1,2}^2 \cdot (\phi(x_1)\phi(x_2) \otimes \phi(x_3)\phi(x_4))$$

Figure 2: Examples of the algebraic representation of graphs in terms of elements of $\mathsf{S}(V)^{\otimes v}$.

Usually Feynman graphs involve edges of different types depending on particle species, e.g. straight for fermions, wiggly for bosons, etc. Thus, suppose that there are m different fields ϕ^a, with $a = 1, \ldots, m$, interacting. Each of these fields is associated with an edge of certain kind. In the present context, this means that edges are represented by distinct elements $R_{i,j}^a$, $R_{i,i}^a \in \mathsf{S}(V)^{\otimes v}$, with $a = 1, \ldots, m$, given by

$$R_{i,j}^a := \int \mathrm{d}x\,\mathrm{d}y\, G_F^{a\,-1}(x,y)\,(\mathbf{1}^{\otimes i-1} \otimes \phi^a(x) \otimes \mathbf{1}^{\otimes j-i-1} \otimes \phi^a(y) \otimes \mathbf{1}^{\otimes v-j})\,,$$

and

$$R_{i,i}^a := \int \mathrm{d}x\,\mathrm{d}y\, G_F^{a\,-1}(x,y)\,(\mathbf{1}^{\otimes i-1} \otimes \phi^a(x)\phi^a(y) \otimes \mathbf{1}^{\otimes v-i})\,,$$

respectively. Therefore, the elements $R_{i,j}$ and $R_{i,i}$ that we consider read explicitly as $R_{i,j} = \sum_{a=1}^m R_{i,j}^a$ and $R_{i,i} = \sum_{a=1}^m R_{i,i}^a$, respectively.

A fundamental property of the algebraic representation is that the ordering of the tensor factors of $\mathsf{S}(V)^{\otimes v}$ induces an ordering of the vertices of the graphs. However, when applying $\nu^{\otimes v}$ the ordering is "forgotten". Indeed, it is not relevant for the interpretation of graphs as Feynman graphs, but only plays a role at the level of their algebraic representation. Moreover, usually the elements of $\mathsf{S}(V)^{\otimes v}$ yield as linear combinations of expressions corresponding to graphs. In this context, we call the scalar multiplying the expression for a given graph the *weight* of the graph. Clearly, if we are interested in unordered graphs, the weight of such a graph is the sum of the weights of all vertex ordered graphs that correspond to it upon forgetting the vertex order.

3 Generating connected graphs via Hopf algebra

Statement of result

We state the main result of [11]. This may be described as an algorithm to recursively generate all connected graphs. In particular, each graph is produced together with a weight factor given by the inverse of its symmetry factor. All graphs are generated in the Hopf algebraic representation introduced in Section 2. This allows their direct evaluation as Feynman graphs.

We notice that the discussion here applies to bare n-point functions only.

The elements $R_{i,j}$ and $R_{i,i}$, given by formulas (1) and (2), respectively, are used to define the following linear maps:

- $T_i : \mathsf{S}(V)^{\otimes v} \to \mathsf{S}(V)^{\otimes v}$, with $1 \le i \le v$, as the operator $R_{i,i}$ together with the factor $1/2$:

$$T_i := \frac{1}{2} R_{i,i}\,,$$

- $Q_i : \mathsf{S}(V)^{\otimes v} \to \mathsf{S}(V)^{\otimes v+1}$, with $1 \le i \le v$, given by the composition of $R_{i,i+1}$ with the coproduct applied to the ith component of $\mathsf{S}(V)^{\otimes v}$, i.e., $\Delta_i := \mathrm{id}^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}^{\otimes v-i} : \mathsf{S}(V)^{\otimes v} \to \mathsf{S}(V)^{\otimes v+1}$, together with the factor $1/2$:

$$Q_i := \frac{1}{2} R_{i,i+1} \circ \Delta_i\,.$$

The map T_i endows the vertex i of a vertex ordered graph with a self-loop. The map Q_i splits the vertex i into two new vertices, numbered i and $i+1$, distributes the ends of edges ending on the split vertex between the two new ones in all possible ways and connects the two new vertices with an edge.

The maps T_i increase both the loop and edge numbers of a graph by one unit, leaving the vertex number invariant. Also, the maps Q_i increase both the vertex and edge numbers by one unit, leaving the loop number invariant.

Clearly, both the maps T_i and Q_i produce connected graphs from connected ones, so that the following theorem holds.

Theorem 1. *Let $l, n \ge 0$, $v \ge 1$ denote integers and let the set of maps $\Omega^{l,v} : \mathsf{S}(V) \to \mathsf{S}(V)^{\otimes v}$ be defined recursively as follows:*

$$\Omega^{0,1} := \mathrm{id}\,,$$

$$\Omega^{l,v} := \frac{1}{l+v-1} \left(\sum_{i=1}^{v-1} Q_i \circ \Omega^{l,v-1} + \sum_{i=1}^{v} T_i \circ \Omega^{l-1,v} \right)\,. \qquad (3)$$

Then, for fixed values of l, v, n and operator labels $x_1,...,x_n$, $\Omega^{l,v}(\phi(x_1)\cdots\phi(x_n))$ corresponds to the weighted sum over all connected graphs with l loops, v vertices and n external edges whose end points are labeled by $x_1,\ldots,x_n$, each with weight the inverse of its symmetry factor.

In the recursion equation above the Q and T summands do not appear when $v = 1$ or when $l = 0$, respectively. The proof of Theorem 1 proceeds by induction on the number of internal edges $e = l + v - 1$ [11]. Moreover, formula (3) is an example of a double recursion. Therefore, its algorithmic implementation is that of any recurrence which makes two calls to itself, such as the defining recurrence of the binomial coefficients.

Now, we turn to the interpretation in terms of Feynman graphs and n-point functions. Denote the l-loop and v-vertex contribution to the ensemble σ of connected n-point functions by $\sigma^{l,v}$. The l-loop order contribution σ^l to σ and σ itself, are given by

$$\sigma^l = \sum_{v=0}^{\infty} \sigma^{l,v}, \qquad \sigma = \sum_{l=0}^{\infty} \sigma^l.$$

There is only one contribution with zero vertex number. This is the Feynman propagator contributing to the 2-point function. Hence, $\sigma^{l,v}$ is zero if $v = 0$ and $l \neq 0$, while $\sigma^{0,0}$ is non-zero only on $V \otimes V$ and coincides there with the Feynman propagator. All non-zero vertex number contributions are captured by the following corollary.

Corollary 2. *For $v \geq 1$:*

$$\sigma^{l,v} = \nu^{\otimes v} \circ \Omega^{l,v}.$$

Alternative recursion formula

A key feature of $\Omega^{l,v}$ is that of satisfying an alternative recursion relation. This has the advantage over (3), that it may be translated directly into a recursion relation of the resulting n-point functions $\sigma^{l,v}$, related via Corollary 2.

Proposition 3. *Let $v \geq 1$ and $l \geq 0$, but not $v = 1$ and $l = 0$. Then,*

$$\Omega^{l,v} = \frac{1}{l+v-1}\left(\Omega^{l-1,v} \circ T + \sum_{a=0}^{l}\sum_{b=1}^{v-1}\left(\Omega^{a,b} \otimes \Omega^{l-a,v-b}\right) \circ Q\right). \qquad (4)$$

The first summand does not contribute if $l = 0$, while the second does not contribute if $v = 1$.

Proposition 3 is proved by induction on the number of internal edges $e = l + v - 1$ [11]. Formula (4) has a straightforward interpretation in terms of sums over weighted graphs following the correspondence of Section 2. Namely, the formula states that the weighted sum over graphs with l loops and v vertices is given by a sum of two terms divided by e. The first term is the sum over all weighted graphs with $l - 1$ loops and v vertices which have an extra internal edge attached, its end points being connected to vertices in all possible ways. The second term is a sum over all ordered pairs of weighted graphs with total number of vertices equal to v and total number of loops equal to l, connected in all possible ways with an internal edge.

Combining this result with Corollary 2 yields the corresponding recursion formula for $\sigma^{l,v}$.

Corollary 4. *Let $v \geq 1$ and $l \geq 0$, but not $v = 1$ and $l = 0$. Then,*

$$\sigma^{l,v} = \frac{1}{l + v - 1} \left(\sigma^{l-1,v} \circ T + \sum_{a=0}^{l} \sum_{b=1}^{v-1} \left(\sigma^{a,b} \otimes \sigma^{l-a,v-b} \right) \circ Q \right).$$

It is understood that the first summand does not contribute if $l = 0$, while the second does not contribute if $v = 1$.

4   Connected and 1PI n-point functions

Now, we turn attention to simple algebraic expressions for the relation between connected and 1PI n-point functions following from Theorem 1 and Corollary 4 [10].

Theorem 5. *The connected n-point functions σ may be expressed in terms of the 1PI ones τ through the formula*

$$\sigma = \sigma^0 + \sum_{v=1}^{\infty} \sigma^v, \quad with \quad \sigma^v := \tau^{\otimes v} \circ \Omega^{0,v},$$

where σ^0 is non-zero only on $V \otimes V$ and coincides there with the Feynman propagator.

Since the 0-point and 1-point 1PI functions are zero, Theorem 5 holds as long as all tree graphs with v vertices, n external edges whose end points are labeled by $x_1, \ldots, x_n$, and the property that each vertex has valence at least two, have no non-trivial symmetries. In other words, all tree graphs with the aforesaid properties are required to occur in $\Omega^{0,v}(\phi(x_1) \cdots \phi(x_n))$ with exactly weight 1. This is ensured by the following lemma.

Lemma 6. *Consider a tree graph γ, all of whose vertices have valence at least two. Then, γ has no non-trivial symmetries.*

In particular, by Corollary 4, the σ^v satisfy the following recursion formula.

Proposition 7. *σ^v may be determined recursively via $\sigma^1 = \tau$ and with the recursion equation for $v \geq 2$,*

$$\sigma^v = \frac{1}{v-1} \sum_{i=1}^{v-1} (\sigma^i \otimes \sigma^{v-i}) \circ Q.$$

We notice that all results immediately carry over to the relation between connected and modified 1PI n-functions. To this end, we modify the definition of R (and consequently that of $\Omega^{0,v}$) by replacing G_F^{-1} with $G_c^{(2)\,-1}$ in formula (1). Recall that for the modified 1PI functions $\hat{G}_{1\text{PI}}$ not only 0- and 1-point functions vanish, but also 2-point functions. This implies that only trees contribute which have the property that all their vertices have valence at least three. Actually, for a given number of external edges, there are only finitely many such trees. Therefore, a connected function yields a finite sum over tree graphs with modified 1PI functions as vertices, for each set of external edges, i.e., for each element of $\mathsf{S}(V)$ to which it is applied.

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